

Sorting and Selection in Posets

Constantinos Daskalakis* Richard M. Karp† Elchanan Mossel‡
 Samantha Riesenfeld§ Elad Verbin¶

July 10, 2007

Abstract

Classical problems of sorting and searching assume an underlying linear ordering of the objects being compared. In this paper, we study a more general setting, in which some pairs of objects are incomparable. This generalization is relevant in applications related to rankings in sports, college admissions, or conference submissions. It also has potential applications in biology, such as comparing the evolutionary fitness of different strains of bacteria, or understanding input-output relations among a set of metabolic reactions or the causal influences among a set of interacting genes or proteins. Our results improve and extend results from two decades ago of Faigle and Turán, who were the first to consider some of the problems considered here.

A poset is defined as a set of elements with a transitive partial order where some pairs of elements may be incomparable. A measure of complexity of a poset is given by its *width*, which is the maximum size of a set of mutually incomparable elements. We consider algorithms that obtain information about a poset by queries that compare two elements. We consider two complexity measures: query complexity, which counts only the number of queries, and total complexity, which counts all operations.

We present an algorithm that sorts a width w poset of size n and has query complexity $O(n(w + \log n))$, which is within a constant factor of the information-theoretic lower bound. We also show that a variant of Mergesort has query complexity $O(wn \log \frac{n}{w})$ and total complexity $O(w^2 n \log \frac{n}{w})$. Faigle and Turán have shown that the sorting problem has query complexity $O(wn \log \frac{n}{w})$ but did not address the total complexity of the problem.

Two problems related to sorting are the problem of finding the minimal elements in a poset and its generalization of finding the bottom k “levels”, called the k -*selection* problem. We give efficient deterministic and randomized algorithms for finding the minimal elements with $O(wn)$ query and total complexity. We provide matching lower bounds for the query complexity up to a factor of 2 and generalize the results to the k -selection problem. We also derive upper bounds on the total complexity of some other problems of a similar flavor, such as computing a linear extension of a poset and computing the heights of all elements.

Many open problems remain, of which the most significant is to determine the precise total complexity of sorting, as well as the precise query and total complexity of k -selection. It would also be interesting to find efficient static and dynamic data structures that play the same role for partial orders that heaps and binary search trees play for total orders.

*Computer Science, U.C. Berkeley. email: costis@cs.berkeley.edu

†Computer Science, U.C. Berkeley. email: karp@icsi.berkeley.edu

‡Statistics, U.C. Berkeley. email: mossel@stat.berkeley.edu. Supported by NSF grants DMS 0528488 and DMS 0548249 (CAREER), ONR and N0014-07-1-05-06 and a Sloan Fellowship in Mathematics

§Computer Science, U.C. Berkeley. email: samr@EECS.Berkeley.EDU

¶Computer Science, Tel Aviv University. email: eladv@post.tau.ac.il

1 Introduction

Sorting is the process of determining the underlying linear ordering of a set S of n elements. *Comparison algorithms*, in which direct comparisons between pairs of elements of S are the only means of acquiring information about the linear ordering, form an important subclass, including such familiar algorithms as Heapsort, Quicksort, Mergesort, Shellsort and Bubblesort.

In this paper we extend the theory of comparison sorting to the case where the underlying structure of the set S is a partial order, in which an element may be larger than, smaller than, or incomparable to another element, and the “larger-than” relation is transitive and irreflexive. Such a set is called a partially ordered set, or poset. This extension is applicable to many ranking problems where certain pairs of elements are incomparable. Examples include ranking college applicants, conference submissions, tennis players, strains of bacteria according to their evolutionary fitness, and points in R^d under the coordinate-wise dominance relation.

Our algorithms gather information by *queries* to an oracle. The oracle’s response to a query involving elements x and y is either the relation between x and y or a statement of their incomparability. In many applications, a query may involve extensive effort (for example, running an experiment to determine the relative evolutionary fitness of two strains of bacteria, or comparing the credentials of two candidates for nomination to a learned society). We therefore consider two measures of complexity for an algorithm or problem: the *query complexity*, which is the number of queries performed, and the *total complexity*, which is the number of computational operations of all types performed (basic operations include standard data structure operations involving one or two elements of the poset).

A partial order on a set can be thought of as the reachability relation of a directed acyclic graph (DAG). More generally, a transitive relation (which is not necessarily irreflexive) can be thought of as the reachability relation of a general directed graph. In applications, the relation represents the direct and indirect causal influences among a set of variables, processes, or components of a system. We show that with negligible overhead, the problem of sorting a transitive relation reduces to the problem of sorting a partial order. Our algorithms thus allow one to reconstruct general directed graphs, given an oracle for queries on reachability from one node to another. As directed graphs are the basic model for many real-life networks including social, information, biological and technological networks (see [14] for a survey), our algorithms provide a potential tool for the reconstruction of such networks.

There is a vast literature on algorithms for determining properties of an initially unknown total order by means of comparisons. Partial orders often arise in these studies as a representation of the “information state” at a general step of such an algorithm. In such cases the incomparability of two elements simply means that their true relation has not been determined yet. The present work is quite different, in that the underlying structure to be discovered is a partial order, and incomparability of elements is inherent, rather than representing temporary lack of information. Nevertheless, the body of work on comparison algorithms for total orders provides valuable tools and insights that can be extended to the present context (e.g. [1, 7, 9, 12, 13]).

The model considered here was previously considered by Faigle and Turán [5], who presented two algorithms for the problem of sorting a partial ordered set, which they term “identification” of a poset. We formally describe their results in Section 1.2. A recent paper [15] considers an extension of the searching and sorting problem to partial orders that are either trees or forests.

1.1 Definitions

To precisely describe the problems considered in this paper and our results, we require some formal definitions. A partially ordered set, or poset, is a pair $\mathcal{P} = (P, \succ)$, where P is a set of elements and $\succ \subset P \times P$ is an irreflexive, transitive binary relation. For elements $a, b \in P$, if $(a, b) \in \succ$, we write $a \succ b$ and we say that a *dominates* b , or that b is *smaller than* a . If $a \not\succ b$ and $b \not\succ a$, we say that a and b are *incomparable* and write $a \not\sim b$.

A *chain* $C \subseteq P$ is a subset of mutually comparable elements, that is, a subset such that for any elements $c_i, c_j \in C$, $i \neq j$, either $c_i \succ c_j$ or $c_j \succ c_i$. An *ideal* $I \subseteq P$ is a subset of elements such that if $x \in I$ and $x \succ y$, then $y \in I$. The *height* of an element a is the maximum cardinality of a chain whose elements are all dominated by a . We call the set $\{a : \forall b, b \succ a \text{ or } b \not\sim a\}$ of elements of height 0 the *minimal* elements. An *anti-chain* $A \subseteq P$ is a subset of mutually incomparable elements. The *width* $w(\mathcal{P})$ of poset \mathcal{P} is defined to be the maximum cardinality of an anti-chain of \mathcal{P} .

A *decomposition* \mathcal{C} of \mathcal{P} into chains is a family $\mathcal{C} = \{C_1, C_2, \dots, C_q\}$ of disjoint chains such that their union is P . The *size* of a decomposition is the number of chains in it. The width $w(\mathcal{P})$ is clearly a lower bound on the size of any decomposition of \mathcal{P} . We make frequent use of *Dilworth's Theorem*, which states that there is a decomposition of \mathcal{P} of size $w(\mathcal{P})$. A decomposition of size $w(\mathcal{P})$ is called a *minimum chain decomposition*.

1.2 Sorting and k -selection

The central computational problems of this paper are *sorting* and *k -selection*. The sorting problem is to completely determine the partial order on a set of n elements, and the k -selection problem is to determine the set of elements of height at most $k - 1$, i.e., the set of elements in the k bottom levels of the partial order. In both problems we are given an upper bound of w on the width of the partial order.

In the absence of a bound on the width, the query complexity of the sorting problem is exactly $\binom{n}{2}$, in view of the worst-case example in which all pairs of elements are incomparable. In the classical sorting and selection problems, $w = 1$. Our interest is mainly in the case where $w \ll n$, since this assumption is natural in many of the applications. Furthermore, if w is of the same order as n , then it is easy to see that the complexity of sorting is of order n^2 , as in the case where no restrictions are imposed on the poset.

Faigle and Turán [5] have described two algorithms for sorting posets, both of which have query complexity $O\left(wn \log \frac{n}{w}\right)$. (In fact the second algorithm is shown to have query complexity $O(n \log N_{\mathcal{P}})$, where $N_{\mathcal{P}}$ is the number of ideals in input poset \mathcal{P} . It is easy to see that $N_{\mathcal{P}} = O(n^w)$ if \mathcal{P} has width w , and that $N_{\mathcal{P}} = (n/w)^w$ if \mathcal{P} consists of w incomparable chains, each of size n/w .) The total complexity of sorting posets has not been considered. However, the total complexity of the first algorithm given by Faigle and Turán depends on the subroutine for computing a chain decomposition (the complexity of which is not analyzed in [5]). It is not clear if there exists a polynomial-time implementation of the second algorithm.

1.3 Techniques

It is natural to approach the problems of sorting and k -selection in posets by considering generalizations of the well-known algorithms for the case of total orders, whose running times are closely matched by proven lower bounds. Somewhat surprisingly, natural generalizations of the classic algorithms *do not* provide optimal poset algorithms in terms of total and query complexity.

In the case of sorting, the generalization of Mergesort considered here loses a factor of w in its total complexity compared to the information-theoretic lower bound. Interestingly, one can achieve the information-theoretic lower bound on query complexity (up to a constant factor) by carefully exploiting the structure of the poset. We do not know whether it is possible to achieve the information-theoretic bound on total complexity.

The seemingly easier problem of k -selection still poses some challenges. In particular, nontrivial arguments are needed to obtain both lower and upper bounds. Moreover, there is a gap of factor 2 between the lower and upper bound, even for the problem of finding minimal elements.

1.4 Main Results and Paper Outline

In Section 2, we briefly discuss an efficient representation of a poset. The representation is of size $O(\omega n)$, and it allows the relation between any two elements to be retrieved in time $O(1)$.

In Sections 3.1 and 3.2, we prove the following main theorems:

Theorem 1. *There exists an algorithm for sorting a poset of width at most w over n elements with optimal query complexity $O(n(\log n + w))$.*

Theorem 2. *A generalization of Mergesort for sorting a poset of width at most w over n elements has query complexity $O(\omega n \log n)$ and total complexity $O(\omega^2 n \log n)$. The algorithm also provides a minimum chain decomposition of the set.*

In Section 4, we consider the k -selection problem of determining the elements of height less than or equal to $k - 1$. We give upper and lower bounds on the query complexity and total complexity of k -selection within deterministic and randomized models of computation. For the case $k = 1$ (finding the minimal elements), we show that the query complexity and total complexity are $\Theta(\omega n)$. The query upper bounds match the query lower bounds up to a factor of 2.

In Section 5, we give a randomized algorithm, based on a generalization of Quicksort, of expected total complexity $O(n(\log n + w))$ for computing a linear extension of a poset. We also give a randomized algorithm of expected total complexity $O(\omega n \log n)$ for computing the heights of all elements in a poset.

Finally, in Section 6, we show that the results on sorting posets generalize to the case when an upper bound on the width is not known and to the case of transitive relations.

1.5 Acknowledgments:

E.M. would like to thank Mike Saks for the reference to the work of Faigle and Turán [5].

2 Representing a poset: the CHAINMERGE data structure

Once the relation between every pair of elements in a poset has been determined, some representation of this information is required, both for output and for use in our algorithms. The simple CHAINMERGE data structure that we describe here supports constant-time look-ups of the relation between any pair of elements. It is built from a chain decomposition of the poset.

Let $\mathcal{C} = \{C_1, \dots, C_q\}$ be a chain decomposition of a poset $\mathcal{P} = (P, \succ)$. $\text{CHAINMERGE}(\mathcal{P}, \mathcal{C})$ stores, for each element $x \in P$, q indices as follows: Let C_i be the chain of \mathcal{C} containing x . The data structure stores the index of x in C_i and, for all j , $1 \leq j \leq q$, $j \neq i$, the index of the largest element of chain C_j that is dominated by x . The performance of the data structure is characterized by the following lemma.

Claim 3. *Given a query oracle for a poset $\mathcal{P} = (P, \succ)$ and a decomposition \mathcal{C} of \mathcal{P} into q chains, building the CHAINMERGE data structure has query complexity at most $2qn$ and total complexity $O(qn)$, where $n = |P|$. Given CHAINMERGE(\mathcal{P}, \mathcal{C}), the relation in \mathcal{P} of any pair of elements can be found in constant time.*

Proof. The indices corresponding to chain C_j that must be stored for the elements in chain C_i can be found in $O(|C_i| + |C_j|)$ time, using $|C_i| + |C_j|$ queries, by simultaneously scanning C_i and C_j . Since each chain is scanned $2q - 1$ times, it follows that the query complexity of CHAINMERGE(\mathcal{P}, \mathcal{C}) is at most $2qn$, and the total complexity is $O(q \cdot \sum_{i=1}^q |C_i|) = O(qn)$.

Let $x, y \in P$, with $x \in C_i$ and $y \in C_j$. The look-up operation works as follows: If $i = j$, we simply do a comparison on the indices of x and y in C_i , as in the case of a total order. If $i \neq j$, then we look up the index of the largest element of C_j that is dominated by x ; this index is greater than (or equal to) the index of y in C_j if and only if $x \succ y$. If $x \not\succ y$, then we look up the index of the largest element of C_i that is dominated by y ; this index is greater than (or equal to) the index of x in C_i if and only if $y \succ x$. If neither $x \succ y$ nor $y \succ x$, then $x \not\sim y$. \square

3 The sorting problem

We address the problem of *sorting a poset*, which is the computational task of producing a representation of a poset $\mathcal{P} = (P, \succ)$, given the set P of n elements, an upper bound of w on the width of \mathcal{P} , and access to an oracle for \mathcal{P} . (See Section 6.1 for a discussion of the case when an upper bound on the width is not known.) An information-theoretic lower bound on the query complexity of sorting is implied by the following theorem of Brightwell and Goodall [2], which provides a lower bound on the number $N_w(n)$ of posets of width at most w on n elements.

Theorem 4. *The number $N_w(n)$ of partially ordered sets of n elements and width at most w satisfies*

$$\frac{n!}{w!} 4^{n(w-1)} n^{-24w(w-1)} \leq N_w(n) \leq n! 4^{n(w-1)} n^{-(w-2)(w-1)/2} w^{w(w-1)/2}.$$

It follows that, for $w = o\left(\frac{n}{\log n}\right)$,

$$\log N_w(n) = \Theta(n \log n + wn).$$

3.1 An optimal sorting algorithm

In this section, we describe a sorting algorithm that has optimal query complexity, i.e. it sorts a poset of width at most w on n elements using $\Theta(n \log n + wn)$ oracle queries. Our algorithm is not necessarily computationally efficient, so in Section 3.2, we consider efficient solutions to the problem.

Before presenting our algorithm, it is worth discussing an intuitive approach that is different from the one we take. For any set of oracle queries and responses, there is a corresponding set of posets, which we call *candidates*, that are the posets consistent with the responses to these queries. A natural sorting algorithm is to find a sequence of oracle queries such that, for each query (or for a positive fraction of the queries), the possible responses to it partition the space of posets that are candidates after the previous queries into three parts, at least two of which are relatively large. Such an algorithm would achieve the information-theoretic lower bound (up to a constant).

For example, the effectiveness of Quicksort for sorting total orders relies on the fact that most of the queries made by the algorithm partition the space of candidate total orders into two parts,

Algorithm POSET–BININSERTIONSORT(\mathcal{P})**input:** a set P , a query oracle for a poset $\mathcal{P} = (P, \succ)$, and upper bound of w on width of \mathcal{P} **output:** a CHAINMERGE data structure for \mathcal{P}

1. $\mathcal{P}' := (\{e\}, \{\})$, where $e \in P$ is some arbitrary element; /* \mathcal{P}' is the current poset */
2. $P' := \{e\}$; $\mathcal{R}' := \{\}$;
3. $U := P \setminus \{e\}$; /* U is the set of elements that have not been inserted */
4. **while** $U \neq \emptyset$
 - a. pick an arbitrary element $e \in U$; /* e is the element that will be inserted in \mathcal{P}' */
 - b. $U := U \setminus \{e\}$;
 - c. find a chain decomposition $\mathcal{C} = \{C_1, C_2, \dots, C_q\}$ of \mathcal{P}' , with $q \leq w$ chains;
 - d. **for** $i = 1, \dots, q$
 - i. let $C_i = \{e_{i1}, \dots, e_{il_i}\}$, where $e_{il_i} \succ \dots \succ e_{i2} \succ e_{i1}$;
 - ii. do binary search on C_i to find smallest element (if any) that dominates e ;
 - iii. do binary search on C_i to find largest element (if any) that is dominated by e ;
 - e. based on results of binary searches, infer all relations of e with elements of P' ;
 - f. add into \mathcal{R}' all the relations of e with the elements of P' ; $P' := P' \cup \{e\}$;
 - g. $\mathcal{P}' = (P', \mathcal{R}')$;
5. find a decomposition \mathcal{C} of \mathcal{P}' ; build CHAINMERGE($\mathcal{P}', \mathcal{C}$) (no additional queries needed);
6. **return** CHAINMERGE($\mathcal{P}', \mathcal{C}$);

Figure 1: pseudo-code for POSET–BININSERTIONSORT

each of relative size of at least $1/4$. Indeed, in the case of total orders, much more is known: for any subset of queries, there is a query that partitions the space of candidate total orders, i.e. linear extensions, into two parts, each of relative size of at least $3/11$ [10].

In the case of width- w posets, however, it could potentially be the case that most queries partition the space into three parts, one of which is much larger than the other two. For example, if the set consists of w incomparable chains, each of size n/w , then a random query has a response of incomparability with probability about $1 - 1/w$. (On an intuitive level, this explains the extra factor of w in the query complexity of our version of Mergesort, given in Section 3.2.) Hence, we resort to more elaborate sorting strategies.

Our optimal algorithm builds upon a basic algorithm that we call POSET–BININSERTIONSORT, which is identical to “Algorithm A” of Faigle and Turán [5]. The algorithm is inspired by the binary insertion-sort algorithm for total orders. Pseudocode for POSET–BININSERTIONSORT is presented in Figure 1. The natural idea behind POSET–BININSERTIONSORT is to sequentially insert elements into a subset of the poset, while maintaining a chain decomposition of the latter into a number of chains equal to the width w of the poset to be constructed. A straightforward implementation of this idea is to perform a binary search on every chain of the decomposition in order to figure out the relationship of the element being inserted with every element of that chain and, ultimately, with all the elements of the current poset. It turns out that this simple algorithm is not optimal; it is off by a factor of w from the optimum. In the rest of this section, we show how to adapt POSET–BININSERTIONSORT to achieve the information-theoretic lower bound.

We begin by analyzing POSET–BININSERTIONSORT.

Lemma 5 (Faigle & Turán [5]). *POSET–BININSERTIONSORT sorts any partial order \mathcal{P} of width at most w on n elements using at most $O(wn \log n)$ oracle queries.*

Proof. The correctness of POSET-BININSERTIONSORT should be clear from its description. (The simple argument showing that Step 4e can be executed based on the information obtained in Step 4d is similar to the proof for the CHAINMERGE data structure in Section 2.) It is not hard to see that the number of oracle queries incurred by POSET-BININSERTIONSORT for inserting each element is $O(w \log n)$ and, therefore, the total number of queries is $O(wn \log n)$. \square

It follows that, as n scales, the number of queries incurred by the algorithm is more by a factor of w than the lower bound. The Achilles' heel of the POSET-BININSERTIONSORT algorithm is in the method of insertion of an element—specifically, in the way the binary searches of Step 4d are performed. In these sequences of queries, no structural properties of \mathcal{P}' are used for deciding which queries to the oracle are more useful than others; in some sense, the binary searches give the same “attention” to queries whose answer would greatly decrease the number of remaining possibilities and those whose answer is not very informative. However, as we discuss earlier in this section, a sorting algorithm that always makes the most informative query is not guaranteed to be optimal.

Our algorithm tries to resolve this dilemma. We suggest a scheme that has the same structure as the POSET-BININSERTIONSORT algorithm but exploits the structure of the already constructed poset \mathcal{P}' in order to amortize the cost of the queries over the insertions. The amortized query cost matches the information-theoretic bound.

The new algorithm, named ENTROPYSORT, modifies the binary searches of Step 4d into weighted binary searches. The weights assigned to the elements satisfy the following property: the number of queries it takes to insert an element into a chain is proportional to the number of candidate posets that will be eliminated after the insertion of the element. In other words, we spend fewer queries for insertions that are not informative and more queries for insertions that are informative. In some sense, this corresponds to an *entropy-weighted binary search*. To define this notion precisely, we use the following definition.

Definition 1. Suppose that $\mathcal{P}' = (P', \mathcal{R}')$ is a poset of width at most w , U a set of elements such that $U \cap P' = \emptyset$, $u \in U$ and $\mathcal{ER}, \mathcal{PR} \subseteq (\{u\} \times P') \cup (P' \times \{u\})$. We say that $\mathcal{P} = (P' \cup U, \mathcal{R})$ is a width w extension of \mathcal{P}' on U conditioned on $(\mathcal{ER}, \mathcal{PR})$, if \mathcal{P} is a poset of width w , $\mathcal{R} \cap (P' \times P') = \mathcal{R}'$ and, moreover, $\mathcal{ER} \subseteq \mathcal{R}$, $\mathcal{R} \cap \mathcal{PR} = \emptyset$. In other words, \mathcal{P} is an extension of \mathcal{P}' on the elements of U which is consistent with \mathcal{P}' , it contains the relations of u to P' given by \mathcal{ER} and does not contain the relations of u to P' given by \mathcal{PR} . The set \mathcal{ER} is then called the set of enforced relations and the set \mathcal{PR} the set of prohibited relations.

We give in Figure 2 the pseudocode of Step 4d' of ENTROPYSORT, which replaces Step 4d of POSET-BININSERTIONSORT.

The correctness of the ENTROPYSORT algorithm follows trivially from the correctness of POSET-BININSERTIONSORT. We prove next that its query complexity is optimal. Recall that $N_w(n)$ denotes the number of partial orders of width w on n elements.

Theorem 6. ENTROPYSORT sorts any partial order \mathcal{P} of width at most w on n elements using at most $2 \log N_w(n) + 4wn = \Theta(n \log n + wn)$ oracle queries. In particular, the query complexity of the algorithm is at most $2n \log n + 8wn + 2w \log w$.

Proof. We first characterize the number of oracle calls required by the weighted binary searches.

Lemma 7 (Weighted Binary Search). For every $j \in \{1, 2, \dots, \ell_i + 1\}$, if e_{ij} is the smallest element of chain C_i which dominates element e ($j = \ell_i + 1$ corresponds to the case where no element of chain C_i dominates e), then j is found after at most $2 \cdot (1 + \log \frac{\mathcal{D}_i}{\mathcal{D}_{ij}})$ oracle queries in Step v. of the algorithm described above.

Step 4d' for Algorithm ENTROPYSORT(\mathcal{P})

4d'. $\mathcal{ER} = \emptyset$; $\mathcal{PR} = \emptyset$;
for $i = 1, \dots, q$
 i. let $C_i = \{e_{i1}, \dots, e_{i\ell_i}\}$, where $e_{i\ell_i} \succ \dots \succ e_{i2} \succ e_{i1}$;
 ii. **for** $j = 1, \dots, \ell_i + 1$
 • set $\mathcal{ER}_j = \{(e_{ik}, e) | j \leq k \leq \ell_i\}$; set $\mathcal{PR}_j = \{(e_{ik}, e) | 1 \leq k < j\}$;
 • compute \mathcal{D}_{ij} , the number of w -extensions of \mathcal{P}' on U ,
 conditioned on $(\mathcal{ER} \cup \mathcal{ER}_j, \mathcal{PR} \cup \mathcal{PR}_j)$;
 /* \mathcal{D}_{ij} represents the number of posets on P consistent with \mathcal{P}' , $(\mathcal{ER}, \mathcal{PR})$,
 in which e_{ij} is the smallest element of chain C_i that dominates e ;
 $j = \ell_i + 1$ corresponds to the case that no element of C_i dominates e ;*/
endfor
 iii. set $\mathcal{D}_i = \sum_{j=1}^{\ell_i+1} \mathcal{D}_{ij}$;
 /* \mathcal{D}_i is equal to the total number of w -extensions of \mathcal{P}' on U
 conditioned on $(\mathcal{ER}, \mathcal{PR})$ */
 iv. partition the unit interval $[0, 1)$ into $\ell_i + 1$ intervals $([b_j, t_j))_{j=1}^{\ell_i+1}$,
 where $b_1 = 0$, $b_j = t_{j-1}$, for all $j \geq 2$,
 and $t_j = (\sum_{j' \leq j} \mathcal{D}_{ij'}) / \mathcal{D}_i$, for all $j \geq 1$.
 /* each interval corresponds to an element of C_i or a “dummy” element $e_{i\ell_i+1}$ */
 v. do binary search on $[0, 1)$ to find smallest element (if any) of C_i that dominates e ;
 /* weighted version of binary search in Step 4dii of POSET-BININSERTIONSORT */
 set $x = 1/2$; $t = 1/4$; $j^* = 0$;
repeat: find j such that $x \in [b_j, t_j)$;
if $(j = \ell_i + 1$ **and** $e_{i,j-1} \not\succeq e)$ **OR** $(e_{ij} \succ e$ **and** $j = 0)$ **OR** $(e_{ij} \succ e$ **and** $e_{i,j-1} \not\succeq e)$
 set $j^* = j$; **break**; /* found smallest element in C_i that dominates e */
else if $(j = \ell_i + 1)$ **OR** $(e_{ij} \succ e)$
 set $x = x - t$; $t = t * 1/2$; /* look below */
else
 set $x = x + t$; $t = t * 1/2$; /* look above */
 vi. e_{ij^*} is the smallest element of chain C_i that dominates e ;
 set $\mathcal{ER} := \mathcal{ER} \cup \mathcal{ER}_{j^*}$ and $\mathcal{PR} := \mathcal{PR} \cup \mathcal{PR}_{j^*}$;
 vii. find the largest element (if any) of chain C_i that is dominated by e ;
 for $j = 0, 1, \dots, \ell_i$,
 compute \mathcal{D}'_{ij} , the number of posets on P consistent with \mathcal{P}' , $(\mathcal{ER}, \mathcal{PR})$,
 in which e_{ij} is the largest element of chain C_i dominated by e ;
 /* $j = 0$ corresponds to case that no element of C_i is dominated by e ; */
 let $\mathcal{D}'_i = \sum_{j=0}^{\ell_i} \mathcal{D}'_{ij}$;
 do the weighted binary search analogous to that of Step v;
 viii. update accordingly the sets \mathcal{ER} and \mathcal{PR} ;
endfor

Figure 2: Algorithm ENTROPYSORT is obtained by substituting Step 4d' above for Step 4d of the pseudo-code in Figure 1 for POSET-BININSERTIONSORT.

Proof. Let $\lambda = \frac{\mathcal{D}_{ij}}{\mathcal{D}_i}$ be the length of the interval that corresponds to e_{ij} . We wish to prove that the number of queries needed to find e_{ij} is at most $2(1 + \lfloor \log \frac{1}{\lambda} \rfloor)$. From the definition of the weighted binary search, we see that if the interval corresponding to e_{ij} contains a point of the form $2^{-r} \cdot m$ in its interior, where r, m are integers, then the search reaches e_{ij} after at most r steps. Now, an interval of length λ must include a point of the form $2^{-r} \cdot m$, where $r = 1 + \lfloor \log \frac{1}{\lambda} \rfloor$, which concludes the proof. \square

It is important to note that the number of queries spent by the weighted binary search is small for uninformative insertions, which correspond to large \mathcal{D}_{ij} 's, and large for informative ones, which correspond to small \mathcal{D}_{ij} 's. Hence, our use of the term entropy-weighted binary search. A parallel of Lemma 7 holds, of course, for finding the largest element of chain C_i dominated by element e .

Suppose now that $P = \{e_1, \dots, e_n\}$, where e_1, e_2, \dots, e_n is the order in which the elements of P are inserted into poset \mathcal{P}' . Also, denote by \mathcal{P}_k the restriction of poset \mathcal{P} onto the set of elements $\{e_1, e_2, \dots, e_k\}$ and by Z_k the number of width w extensions of poset \mathcal{P}_k on $P \setminus \{e_1, \dots, e_k\}$ conditioned on (\emptyset, \emptyset) . Clearly, $Z_0 \equiv N_w(n)$ and $Z_n = 1$. The following lemma is sufficient to establish the optimality of ENTROPYSORT.

Lemma 8. ENTROPYSORT needs at most $4w + 2 \log \frac{Z_k}{Z_{k+1}}$ oracle queries to insert element e_{k+1} into poset \mathcal{P}_k in order to obtain \mathcal{P}_{k+1} .

Proof. Let $\mathcal{C} = \{C_1, \dots, C_q\}$ be the chain decomposition of the poset \mathcal{P}_k constructed at Step 4c of ENTROPYSORT in the iteration of the algorithm in which element e_{k+1} needs to be inserted into poset \mathcal{P}_k . Suppose also that, for all $i \in \{1, \dots, q\}$, $\pi_i \in \{1, \dots, \ell_i + 1\}$ and $\kappa_i \in \{0, 1, \dots, \ell_i\}$ are the indices computed by the binary searches of Steps v. and vii. of the algorithm. Also, let $\mathcal{D}_i, \mathcal{D}_{ij}, j \in \{1, \dots, \ell_i + 1\}$, and $\mathcal{D}'_i, \mathcal{D}'_{ij}, j \in \{0, \dots, \ell_i\}$, be the quantities computed at Steps ii., iii. and vii. It is not hard to see that the following are satisfied

$$\begin{aligned} Z_k &= \mathcal{D}_1 & \mathcal{D}'_{q\kappa_q} &= Z_{k+1} \\ \mathcal{D}_{i\pi_i} &= \mathcal{D}'_i, \forall i = 1, \dots, q & \mathcal{D}'_{i\kappa_i} &= \mathcal{D}_{i+1}, \forall i = 1, \dots, q-1 \end{aligned}$$

Now, using Lemma 7, it follows that the total number of queries required to construct \mathcal{P}_{k+1} from \mathcal{P}_k is at most

$$\sum_{i=1}^q \left(2 + 2 \log \frac{\mathcal{D}_i}{\mathcal{D}_{i\pi_i}} + 2 + 2 \log \frac{\mathcal{D}'_i}{\mathcal{D}'_{i\kappa_i}} \right) \leq 4w + 2 \log \frac{Z_k}{Z_{k+1}}.$$

\square

Using Lemma 8, the query complexity of ENTROPYSORT is

$$\begin{aligned} \sum_{k=0}^{n-1} (\# \text{ queries needed to insert element } e_{k+1}) &= \sum_{k=0}^{n-1} \left(4w + 2 \log \frac{Z_k}{Z_{k+1}} \right) \\ &= 4wn + 2 \log \frac{Z_0}{Z_n} = 4wn + 2 \log N_w(n). \end{aligned}$$

Taking the logarithm of the upper bound in Theorem 4, it follows that the number of queries required by the algorithm is $2n \log n + 8wn + 2w \log w$.

\square

Algorithm POSET-MERGESORT(\mathcal{P})**input:** a set P , a query oracle for a poset $\mathcal{P} = (P, \succ)$, and upper bound w on width of \mathcal{P} **output:** a CHAINMERGE data structure for \mathcal{P}

run POSET-MERGESORT-RECURSE(P) producing a decomposition \mathcal{C} of \mathcal{P} into w chains;
 build and **return** CHAINMERGE(\mathcal{P}, \mathcal{C});

Procedure POSET-MERGESORT-RECURSE(P')**input:** a subset $P' \subseteq P$, a query oracle for $\mathcal{P} = (P, \succ)$, an upper bound w on the width of \mathcal{P} **output:** a decomposition into at most w chains of the poset \mathcal{P}' induced by \succ on P' **if** $|P'| \leq w$ **then return** the trivial decomposition of \mathcal{P}' into chains of length 1**else**

1. partition P' into two parts of equal size, P'_1 and P'_2 ;
 2. run POSET-MERGESORT-RECURSE(P'_1) and POSET-MERGESORT-RECURSE(P'_2);
 3. collect the outputs to get a decomposition \mathcal{C} of \mathcal{P}' into $q \leq 2w$ chains;
 4. if $q > w$, run PEELING(\mathcal{P}, \mathcal{C}), to get a decomposition \mathcal{C}' of \mathcal{P}' into w chains;
- return**
- \mathcal{C}'
- ;

Figure 3: pseudo-code for POSET-MERGESORT

3.2 An efficient sorting algorithm

In this section, we turn to the problem of efficient sorting. Superficially, the POSET-MERGESORT algorithm that we present has a recursive structure that is similar to the classical Mergesort algorithm. The merge step is quite different, however; it makes crucial use of the technical PEELING algorithm in order to efficiently maintain a small chain decomposition of the poset throughout the recursion. The PEELING algorithm, described formally in Section 3.2.2, is a specialization of the classic flow-based bipartite-matching algorithm [6] that is efficient in the comparison model.

3.2.1 Algorithm POSET-MERGESORT

Given a set P , a query oracle for a poset $\mathcal{P} = (P, \succ)$, and an upper bound of w on the width of \mathcal{P} , the POSET-MERGESORT algorithm produces a decomposition of \mathcal{P} into w chains and concludes by building a CHAINMERGE data structure. To get the chain decomposition, the algorithm partitions the elements of P arbitrarily into two subsets of (as close as possible to) equal size; it then finds a chain decomposition of each subset recursively. The recursive call returns a decomposition of each subset into at most w chains, which constitutes a decomposition of the whole set P into at most $2w$ chains. Then the PEELING algorithm of Section 3.2.2 is applied to reduce the decomposition to a decomposition of w chains. Given a decomposition of $P' \subseteq P$, where $m = |P'|$, into at most $2w$ chains, the PEELING algorithm returns a decomposition of P' into w chains using $2wm$ queries and $O(w^2m)$ time. Figure 3 shows pseudo-code for POSET-MERGESORT.

Theorem 9. POSET-MERGESORT sorts any poset \mathcal{P} of width at most w on n elements using at most $2wn \log(n/w)$ queries, with total complexity $O(w^2n \log(n/w))$.

Proof. The correctness of POSET-MERGESORT is immediate. Let $T(m)$ and $Q(m)$ be the worst-case total and query complexity, respectively, of the procedure POSET-MERGESORT-RECURSE on a poset of width w containing m elements. When $m \leq w$, $T(m) = O(w)$ and $Q(m) = 0$. When $m > w$, $T(m) = 2T(m/2) + O(w^2m)$ and $Q(m) \leq 2Q(m/2) + 2wm$. Therefore, $T(n) = O(w^2n \log(n/w))$ and $Q(n) \leq 2wn \log(n/w)$. The cost incurred by the last step of the algorithm, i.e. that of building the CHAINMERGE, is negligible. \square

3.2.2 The PEELING algorithm

In this section we present an algorithm that efficiently reduces the size of a given decomposition of a poset. It can be seen as an adaptation of the classic flow-based bipartite-matching algorithm [6] that is designed to be efficient in the oracle model. The PEELING algorithm is given an oracle for poset $\mathcal{P} = (P, \succ)$, where $n = |P|$, and a decomposition of P into at most $2w$ chains. It first builds a CHAINMERGE data structure using at most $2qn$ queries and time $O(qn)$. Every query the algorithm makes after that is actually a look-up in the data structure and therefore takes constant time and no oracle call.

The PEELING algorithm proceeds in a number of *peeling iterations*. Each iteration produces a decomposition of \mathcal{P} with one less chain, until after at most w peeling iterations, a decomposition of \mathcal{P} into w chains is obtained. A detailed formal description of the algorithm is given in Figure 4.

Theorem 10. *Given an oracle for $\mathcal{P} = (P, \succ)$, where $n = |P|$, and a decomposition of \mathcal{P} into at most $2w$ chains, the PEELING algorithm returns a decomposition of \mathcal{P} into w chains. It has query complexity at most $2wn$ and total complexity $O(w^2n)$.*

Proof. To prove the correctness of one peeling iteration, we first observe that it is always possible to find a pair (x, y) of top elements such that $y \succ x$, as specified in Step 1a, since the size of any anti-chain is at most the width of \mathcal{P} , which is less than the number of chains in the decomposition. We now argue that it is possible to find a subsequence of dislodgements as specified by Step 2a. Let y_t be the element defined in step 3 of the algorithm. Since y_t was dislodged by x_t , x_t was the top element of some list when that happened. In order for x_t to be a top element, it was either top from the beginning, or its parent y_{t-1} must have been dislodged by some element x_{t-1} , and so on.

We claim that, given a decomposition into q chains, one peeling iteration produces a decomposition of \mathcal{P} into $q - 1$ chains. Recall that $y_1 \succ x_1$ and, moreover, for every i , $2 \leq i \leq t$, $y_i \succ x_i$, and $y_{i-1} \succ x_i$. Observe that after Step 4 of the peeling iteration, the total number of pointers has increased by 1. Therefore, if the link structure remains a union of disconnected chains, the number of chains must have decreased by 1, since 1 extra pointer implies 1 less chain. It can be seen that the switches performed by Step 4 of the algorithm maintain the invariant that the in-degree and out-degree of every vertex is bounded by 1. Moreover, no cycles are introduced since every pointer that is added corresponds to a valid relation. Therefore, the link structure is indeed a union of disconnected chains.

The query complexity of the PEELING algorithm is exactly the query complexity of CHAINMERGE, which is $2wn$. We show next that one peeling iteration can be implemented in time $O(qn)$, which implies the claim.

In order to implement one peeling iteration in time $O(qn)$, a little book-keeping is needed, in particular, for Step 2a. We maintain during the peeling iteration a list L of potentially-comparable pairs of elements. At any time, if a pair (x, y) is in L , then x and y are top elements. At the beginning of the iteration, L consists of all pairs (x, y) where x and y are top elements. Any time an element x that was not a top element becomes a top element, we add to L the set of all

```

Algorithm PEELING( $\mathcal{P}, \mathcal{C}$ )
: a query oracle for poset  $\mathcal{P} = (P, \succ)$ , an upper bound of  $w$  on the width of  $\mathcal{P}$ ,
        and a decomposition  $C = \{C_1, \dots, C_q\}$  of  $\mathcal{P}$ , where  $q \leq 2w$ 
: a decomposition of  $\mathcal{P}$  into  $w$  chains

    build CHAINMERGE( $\mathcal{P}, \mathcal{C}$ ); /* All further queries are look-ups. */
    for  $i = 1, \dots, q$ 
        construct a linked list for each chain  $C_i = e_{il_i} \rightarrow \dots \rightarrow e_{i2} \rightarrow e_{i1}$ ,
        where  $e_{il_i} \succ \dots \succ e_{i2} \succ e_{i1}$ ;
    while  $q > w$ , perform a peeling iteration:
        1. for  $i = 1, \dots, q$ , set  $C'_i = C_i$ ;
        2. while every  $C'_i$  is nonempty
            /* the largest element of each  $C'_i$  is a top element */
            a. find a pair  $(x, y)$ ,  $x \in C'_i$ ,  $y \in C'_j$ , of top elements such that  $y \succ x$ ;
            b. delete  $y$  from  $C'_j$ ; /*  $x$  dislodges  $y$  */
        3. in sequence of dislodgements, find subsequence  $(x_1, y_1), \dots, (x_t, y_t)$  such that:
            •  $y_t$  is the element whose deletion (in step 2b) created an empty chain;
            • for  $i = 2, \dots, t$ ,  $y_{i-1}$  is the parent of  $x_i$  in its original chain;
            •  $x_1$  is the top element of one of the original chains;
        4. modify the original chains  $C_1, \dots, C_q$ :
            a. for  $i = 2, \dots, t$ 
                i. delete the pointer going from  $y_{i-1}$  to  $x_i$ ;
                ii. replace it with a pointer going from  $y_i$  to  $x_i$ ;
            b. add a pointer going from  $y_1$  to  $x_1$ ;
        5. set  $q = q - 1$ , and re-index the modified original chains from 1 to  $q - 1$ ;
    return the current chain decomposition, containing  $w$  chains

```

Figure 4: pseudo-code for the PEELING Algorithm

pairs (x, y) such that y is currently a top element. Whenever a top element x is dislodged, we remove from L all pairs that contain x . When Step 2a requires us to find a pair of comparable top elements, we take an arbitrary pair (x, y) out of L and check if x and y are comparable. If they are not comparable, we remove (x, y) from L , and try the next pair. Thus, we never compare a pair of top elements more than once. Since each element of P is responsible for inserting at most q pairs to L (when it becomes a top element), it follows that a peeling iteration can be implemented in time $O(qn)$. □

4 The k -selection problem

The k -selection problem is the natural problem of finding the elements in the bottom k layers, i.e., the elements of height at most $k - 1$, of a poset $\mathcal{P} = (P, \succ)$, given the set P of n elements, an upper bound w on the width of \mathcal{P} , and a query oracle for \mathcal{P} . We present upper and lower bounds on the query and total complexity of k -selection, both for deterministic and randomized computational models, for the special case of $k = 1$ as well as the general version. While our upper bounds arise from natural generalizations of analogous algorithms for total orders, the lower bounds are achieved quite differently. We conjecture that our deterministic lower bound for the case of $k = 1$ is actually tight, though the upper bound is off by a factor of 2.

4.1 Upper bounds

In this section we analyze some deterministic and randomized algorithms for the k -selection problem. We begin with the 1-selection problem, i.e., the problem of finding the minimal elements.

Theorem 11. *The minimal elements can be found deterministically with at most wn queries and $O(wn)$ total complexity.*

Proof. The algorithm updates a set of size w of elements that are candidates for being smallest elements. Initialize $T_0 = \emptyset$. Assume that the elements are x_1, \dots, x_n . At step t :

- Compare x_t to all elements in T_{t-1} .
- If there exists some $a \in T_{t-1}$ such that $x_t \succ a$, do nothing.
- Otherwise, remove from T_{t-1} all elements a such that $a \succ x_t$ and put x_t into T_t .

At the termination of the algorithm, the set T_n contains all height 0 elements. By construction of T_t , for all t , the elements in T_t are mutually incomparable. Therefore, for all t , it holds that $|T_t| \leq w$, and hence the query complexity of the algorithm is at most wn . □

Theorem 12. *There exists a randomized algorithm that finds the minimal elements in an expected number of queries that is upper bounded by*

$$\frac{w+1}{2}n + \frac{w^2-w}{2}(\log n - \log w).$$

Proof. The algorithm is similar to the algorithm for the proof of Theorem 11, with modifications to avoid (in expectation) worst-case behavior. Let σ be a permutation of $[n]$ chosen uniformly at random. Let $T_1 = \{x_{\sigma(1)}\}$. For $1 \leq t < n$, at step t :

- Let i be an index of the candidates in T_{t-1} , i.e. $T_{t-1} = \{x_{i(1)}, \dots, x_{i(r)}\}$, where $r \leq w$.
- Let $T_t = T_{t-1}$. Let τ be a permutation of $[r]$ chosen uniformly at random.
- For $j = 1, \dots, r$:
 - If $x_{\sigma(t)} \succ x_{i(\tau(j))}$, exit the loop and move to step $t + 1$.
 - If $x_{i(\tau(j))} \succ x_{\sigma(t)}$, remove $x_{i(\tau(j))}$ from T_t .
- Add $x_{\sigma(t)}$ to T_t .

As in the previous algorithm, it is easy to see that at each step t , the set T_t contains all the minimal elements of $A_t = \{x_{\sigma(1)}, \dots, x_{\sigma(t)}\}$ and that $|T_t| \leq w$. Note furthermore that at step t ,

$$\mathbf{P}[x_{\sigma(t)} \text{ is minimal for } A_t] \leq \frac{w}{t}.$$

If $x_{\sigma(t)}$ is not minimal for A_t , then the expected number of comparisons needed until $x_{\sigma(t)}$ is compared to an element $a \in A_t$ that dominates $x_{\sigma(t)}$ is clearly at most $(w+1)/2$. We thus conclude that the expected running time of the algorithm is bounded by:

$$\begin{aligned} \sum_{t=2}^w (t-1) + \sum_{t=w+1}^n \left(\frac{w}{t}w + \frac{(t-w)(w+1)}{t} \right) &= \binom{w}{2} + \sum_{t=w+1}^n \frac{1}{2t} (w^2 - w + tw + t) \\ &\leq \frac{w+1}{2}n + \frac{w^2 - w}{2}(\log n - \log w) \end{aligned}$$

□

We now turn to the k -selection problem for $k > 1$. We first provide deterministic upper bounds on query and total complexity.

Theorem 13. *The query complexity of the k -selection problem is at most*

$$16wn + 4n \log(2k) + 6n \log w.$$

Moreover, there exists an efficient k -selection algorithm with query complexity at most

$$8wn \log(2k)$$

and total complexity

$$O(w^2n \log(2k)).$$

Proof. The basic idea is to use the sorting algorithm presented in previous sections in order to update a set of candidates for the k -selection problem. Denote the elements by x_1, \dots, x_n . Let $C_0 = \emptyset$. The algorithm proceeds as follows, beginning with $t=1$:

- While $(t-1)wk + 1 \leq n$, let $D_t = C_{t-1} \cup \{x_{(t-1)wk+1}, \dots, x_{\min(twk, n)}\}$.
- Sort D_t . Let C_t be the solution of the k -selection problem for D_t .

Clearly, at the end of the execution, the last C_t will contain the solution to the k -selection problem. As we have shown, the query complexity of sorting D_t is $4wk \log(2wk) + 16w^2k + 2w \log w$ and, therefore, the query complexity of the algorithm is $\frac{n}{wk}(4wk \log(2wk) + 16w^2k + 2w \log w) = 4n \log(2wk) + 16wn + \frac{2n}{k} \log w$. This proves the first result. Using the computationally efficient sorting algorithm, we have sorting query complexity $8w^2k \log(2k)$ which results in total query complexity $8nw \log(2k)$ and total complexity $O(nw^2 \log(2k))$. □

Next we outline a randomized algorithm with a better coefficient of the main term wn .

Theorem 14. *The k -selection problem has a randomized query complexity of at most*

$$wn + 16kw^2 \log n \log(2k)$$

and total complexity

$$O(wn + \text{poly}(k, w) \log n).$$

Proof. We use the following algorithm:

- Choose an ordering x_1, \dots, x_n of the elements uniformly at random.
- Let $C_{wk} = \{x_1, \dots, x_{wk}\}$ and $D_{wk} = \emptyset$.
- Sort C_{wk} . Remove any elements from C_{wk} that are of height greater than $k - 1$.
- Let $t = wk + 1$. While $t \leq n$ do:
 - Let $C_t = C_{t-1}$ and $D_t = D_{t-1}$.
 - Compare x_t to the maximal elements in C_t in a random order.
 - * For each maximal element $a \in C_t$: if $\text{height}(a) = k - 1$ and $a \succ x_t$, or if $\text{height}(a) < k - 1$ and $x_t \succ a$, then add x_t to D_t , and exit this loop.
 - * If for all elements $a \in C_t$, $x_t \not\succeq a$, then add x_t to D_t and exit this loop;
 - If $|D_t| = wk$ or $t = n$:
 - * Sort $C_t \cup D_t$.
 - * Set C_t to be the elements of height at most $k - 1$ in $C_t \cup D_t$.
 - * Set $D_t = \emptyset$.
- Output the elements of C_n .

It is clear that C_n contains the solution to the k -selection problem. To analyze the query complexity of the algorithm, recall from Theorem 9 that $s(w, k) = 8w^2k \log(2k)$ is an upper bound on the number of queries used by the efficient sorting algorithm to sort $2wk$ elements in a width- w poset.

There are two types of contributions to the number of queries made by the algorithm: (1) comparing elements to the maximal elements of C_t , and (2) sorting the sets C_0 and $C_t \cup D_t$.

To bound the expected number of queries of the first type, we note that for $t \geq kw + 1$, since the elements are in a random order, the probability that x_t ends up in D_t is at most $\min\left(1, \frac{2kw}{t}\right)$. If x_t is not going to be in D_t , then the number of queries needed to verify this is bounded by w . Overall, the expected number of queries needed for comparisons to maximal elements is bounded by wn .

To calculate the expected number of queries of the second type, we bound the expected number of elements that need to be sorted as follows:

$$\sum_{t=kw+1}^n \min\left(1, \frac{2kw}{t}\right) \leq 2kw(\log n - 1).$$

We thus obtain that the total query complexity is bounded above by $wn + 2s(w, k) \log n$. □

4.2 Lower bounds

We obtain lower bounds for the k -selection problem both for adaptive and non-adaptive adversaries. Some of our proofs use the following lower bound on finding the k -th smallest element of a total order on n elements:

Theorem 15 (Fussenegger-Gabow [8]). *The number of queries required to find the k th smallest element of an n -element total order is at least $n - k + \log \binom{n}{k-1}$.*

The proof of Theorem 15 shows that every comparison tree that identifies the k th smallest element must have at least $2^{n-k} \binom{n}{k-1}$ leaves, which implies that the theorem also holds for randomized algorithms.

4.2.1 Adversarial lower bounds

We consider adversarial lower bounds for the k -selection problem. In this model, an adversary simulates the oracle and is allowed to choose her response to a query after she receives it. Any response is legal as long as there is some partial order of width w with which all of her responses are consistent. We begin with the case of $k = 1$, i.e. finding the set of minimal elements.

Theorem 16. *In the adversarial model, at least $\frac{w+1}{2}n - w$ comparisons are needed in order to find the minimal elements.*

Proof. Consider the following adversarial algorithm. The algorithm outputs query responses that correspond to a poset \mathcal{P} of w disjoint chains. Given a query $q(a, b)$, the algorithm outputs a response to the query, and in some cases, it may also announce for one or both of a and b to which chain the element belongs. Note that receiving this extra information can only make things easier for the query algorithm. During the course of the algorithm, the adversary maintains a graph $G = (P, E)$. Whenever the adversary responds that $a \not\sim b$, it adds an edge (a, b) to E .

Let $q_t(a)$ be the number of queries that involve element a , out of the first t queries overall. Let $c(a)$ be the chain assignment that the adversary has announced for element a . (We set $c(a)$ to be undefined for all a , initially.) Let $\{x_i\}_{i=1}^n$ be an indexing, chosen by the adversary, of the elements of P . Let $q(a, b)$ be the t 'th query. The adversary follows the following protocol:

- If $q_t(a) \leq w - 1$ or $q_t(b) \leq w - 1$, return $a \not\sim b$. In addition:
 - If $q_t(a) = w - 1$, choose a chain $c(a)$ for a that is different from all the chains to which a 's neighbors in G belong, and output it.
 - If $q_t(b) = w - 1$ choose a chain $c(b)$ for b that is different from all the chains to which b 's neighbors in G belong, and output it.
- If $q_t(a) > w - 1$, $q_t(b) > w - 1$, and $c(a) \neq c(b)$, then output $a \not\sim b$.
- Otherwise, let i and j be the indices of a and b , respectively (i.e. $a = x_i$ and $b = x_j$). If $i > j$, then output $a \succ b$; otherwise, output $b \succ a$.

It is easy to see that the output of the algorithm is consistent with a width- w poset consisting of w chains that are pairwise incomparable. We will also require that each of the chains is chosen at least once (this is easily achieved).

We now prove a lower bound on the number of queries to this algorithm required to find a proof that the minimal elements are indeed the minimal elements.

In any proof that a is *not* a smallest element, it must be shown to dominate at least one other element, but to get such a response from the adversary, a must be queried against at least $w - 1$ other elements with which it is incomparable. To prove that a minimal element of one chain is indeed minimal, it must be queried at least against the minimal elements of the other chains to rule out the possibility it dominates one of them. Therefore, each element must be compared to at least $w - 1$ elements that are incomparable to it. So the total number of queries of type $q(a, b)$, where $a \not\sim b$, is at least $\frac{w-1}{2}n$.

In addition, for each chain c_i of length n_i , the output must provide a proof of minimality for the minimal element of that chain. By Theorem 15, this contributes $n_i - 1$ comparisons for each chain c_i .

Summing over all the bounds proves the claim. \square

Theorem 17. *Let $r = \frac{n}{2w-1}$. If $k \leq r$ then the number of queries required to solve the k -selection problem is at least*

$$\frac{(w+1)n}{2} - w(k + \log k) - \frac{w^3}{8} + \min \left((w-1) \log \binom{r}{k-1} + \log \binom{rw}{k-1}, \frac{n(r-k)(w-1)}{2r} + \log \binom{n-(w-1)k}{k-1} \right).$$

Proof. The adversarial algorithm outputs query responses exactly as in the proof of Theorem 16, except in the case where the t th query is (a, b) and $q_t(a) = w - 1$ or $q_t(b) = w - 1$. In that case it uses a more specific rule for the assignment of one or both of these elements to chains.

In addition to assigning the elements to chains, the process must also select the k smallest elements in each chain, and the Fussenegger-Gabow theorem (Theorem 15) gives a lower bound, in terms of the lengths of the chains, on the number of comparisons required to do so.

We think of the assignment of elements to chains as a coloring of the elements with w colors. The specific color assignment rule is designed to ensure that, if the number of elements with color c is small, then there must have been many queries in which the element being colored could not receive color c because it had already been declared incomparable to an element with color c . It will then follow that there have been a large number of queries in which an element was declared incomparable to an element with color c . Thus, if many of the chains are very short, then the number of pairs declared incomparable must be very large. On the other hand, if few of the chains are very short, then we can employ the Fussenegger-Gabow Theorem to show that the number of comparisons required to select the k smallest elements in each chain must be large. We obtain the overall lower bound by playing off these two observations against each other.

The color assignment rule is based on a function $d_t(c)$, referred to as the *deviation* of color c after query t , and satisfying the initial condition $d_0(c) = 0$ for all c . The rule is: “assign the eligible color with smallest deviation.”

More specifically, let the t th query be (a_t, b_t) . The adversary processes a_t and then b_t . Recall that $q_t(a)$ is the number of queries involving element a out of the first t queries overall. Element $e \in \{a_t, b_t\}$ is processed exactly as in the proof of Theorem 16 except when $q_t(e) = w - 1$. In that case, let $S_t(e)$ be the set of colors that are *not* currently assigned to neighbors of e ; i.e., the set of colors eligible to be assigned to element e . Let $c^* = \operatorname{argmin}_{c \in S_t(e)} d_{t-1}(c)$. The adversary assigns color c^* to e . Then the deviations of all colors are updated as follows:

1. if $c \notin S_t(e)$ then $d_t(c) = d_{t-1}(c)$;
2. $d_t(c^*) \leftarrow d_{t-1}(c^*) + 1 - \frac{1}{|S_t(e)|}$;

3. For $c \in S_t(e) \setminus \{c^*\}$, $d_t(c) \leftarrow d_{t-1}(c) - \frac{1}{|S_t(e)|}$.

The function $d_t(c)$ has the following interpretation: over the history of the color assignment process, certain steps occur where the adversary has the choice of whether to assign color c to some element; $d_t(c)$ represents the number of times that color c was chosen up to step t , minus the expected number of times it would have been chosen if the same choices had been available at all steps and the color had been chosen uniformly at random from the set of eligible colors.

Because the smallest of the deviations of eligible colors is augmented at each step, it is not possible for any deviation to drift far from zero. Specifically, it can be shown by induction on t that at every step t the sum of the deviations is zero and for $m = 1, 2, \dots, w$, the sum of the m smallest deviations is greater than or equal to $\frac{m(m-w)}{2}$.

Let $\deg_G(a)$ be the degree of a in G at the end of the process. At the end of the process every element of degree greater than or equal to $w - 1$ in G has been assigned to a chain. Each element of degree less than $w - 1$ has not been assigned to a chain, and is therefore called *unassigned*. An unassigned element is called *eligible* for chain c if it has not been compared (and found incomparable) with any element of chain c . Let $s(c)$ be the length of chain c and define $\text{def}(c)$, the *deficiency* of chain c , as $\max(0, k - s(c))$. Define the *total deficiency* DEF as the sum of the deficiencies of all chains.

Let u be the number of unassigned elements. Upon the termination of the process it must be possible to infer from the results of the queries that every unassigned element is of height at most $k - 1$. This implies that, if unassigned element x is eligible for chain c , then the number of unassigned elements eligible for chain c must be at most $\text{def}(c)$. Thus the number of pairs (a, c) such that unassigned element a is eligible for chain c is DEF. Define the *deficiency* of unassigned element a as $w - 1 - \deg_G(a)$. Then the sum of the deficiencies of the unassigned elements is bounded above by DEF, and therefore the sum of the degrees in G of the unassigned elements is at least $(w - 1)u - \text{DEF}$.

By Theorem 15, if $s(c) > k$, then at least $\left(s(c) - k + \log \binom{s(c)}{k-1}\right)$ comparisons are needed to determine the k smallest elements of chain c .

The total number of comparisons is the number of edges that have been placed in G in the course of the algorithm (i.e., the number of pairs that have been declared incomparable by the adversary), plus the number of comparisons required to perform k -selection in each chain. The total number of pairs that have been declared incomparable is $\frac{1}{2} \sum_a \deg_G(a)$.

Let $d(c)$ be the deviation of color c at the end of the process. Let $r(c)$ be the number of steps in the course of the process at which the element being colored was eligible to receive color c . If, at each such step, the color had been chosen uniformly from the set of eligible colors, then the chance of choosing color c would have been at least $\frac{1}{w}$. Thus, by the interpretation of the function $d_t(c)$ given above, $s(c) \geq \frac{r(c)}{w} + d(c)$; equivalently, $r(c) \leq w(s(c) - d(c))$. Also, $\sum_{a|c(a)=c} \deg_G(a) \geq n - r(c) \geq n - w(s(c) - d(c))$. This sum is also at least $(w - 1)s(c)$, since every element assigned to c has been declared incomparable with at least $(w - 1)$ other elements.

We can now combine these observations to obtain our lower bound. For each chain c define $\text{cost}(c) = \frac{1}{2} \sum_{a|c(a)=c} \deg_G(a) + \max\left(0, s(c) - k + \log \binom{s(c)}{k-1}\right)$. Then $\sum_c \text{cost}(c) + \frac{1}{2} ((w - 1)u - \text{DEF})$ is a lower bound on the total number of comparisons, and

$$\begin{aligned} \sum_c \text{cost}(c) &\geq \frac{1}{2} \sum_c \max((w - 1)s(c), n - w(s(c) - d(c))) \\ &\quad + \sum_{c|s(c)>k} \left(s(c) - k + \log \binom{s(c)}{k-1}\right). \end{aligned}$$

To obtain our lower bound we shall minimize this function over all choices of nonnegative integers $s(c)$, u and DEF such that $\sum_c s(c) + u = n$ and $\text{DEF} = \sum_c \max(0, k - s(c))$. Noting that $\sum_c \min(d(c), 0) \geq \min_m m(m - w)/2 = -w^2/8$, we obtain the following lower bound on the total number of comparisons:

$$\frac{(w-1)n}{2} - \frac{\text{DEF}}{2} - \frac{w^3}{8} + \frac{1}{2} \sum_c \max(0, n - (2w-1)s(c)) + \sum_{c|s(c)>k} \left(s(c) - k + \log \binom{s(c)}{k-1} \right) \quad (1)$$

We now restrict attention to the case $k \leq \frac{n}{2w-1}$. Let $r = \frac{n}{2w-1}$. We shall show that, at any global minimum of (1), $\text{DEF} = 0$. To see this, consider any choice of $\{s(c)\}$ such that $\text{DEF} > 0$. Let c be a chain such that $\text{def}(c) > 0$. If $s(c)$ is increased by 1, then DEF decreases by 1, and the net change in the value of quantity (1) is $1 - w$, which is negative.

Thus, in minimizing (1) we may assume that $\text{DEF} = 0$, and hence that $\sum_c s(c) = n$. So (1) may be rewritten as

$$\frac{(w-1)n}{2} - \frac{w^3}{8} + \sum_c F(s(c))$$

where

$$F(s) = \begin{cases} \frac{1}{2} \max(0, n - (2w-1)s) & \text{if } s \leq k \\ \frac{1}{2} \max(0, n - (2w-1)s) + \left(s - k + \log \binom{s}{k-1} \right) & \text{if } s > k. \end{cases}$$

Thus, we have the following minimization problem:

$$\text{Minimize } \sum_c F(s(c)), \text{ subject to } s(c) \geq 0 \text{ and } \sum_c s(c) = n.$$

First, we note that $\sum_{c|k < s(c)} (s(c) - k) = n - wk$. To determine the minimum we consider three ranges of values: the low range $s = k$, medium range $k < s(c) \leq r$, and high range $r < s(c) \leq n$. Observing that $F(s)$ is strictly concave in the medium range, and concave and strictly increasing in the high range, it follows that, at the global minimum of (1), $s(c)$ is equal to either k or r except for one value in the high range and possibly one value strictly within the medium range. The value in the high range is at least rw , since the sum of the values in the low and medium ranges does not exceed $r(w-1)$. If $\sum_{c|k \leq s(c) \leq r} s(c) = (w-1)r - D$, then the unique value of $s(c)$ in the high range is $rw + D$. Moreover, exploiting the concavity of $F(s)$ in the medium range, we claim that $\sum_{c|k \leq s(c) \leq r} \log \binom{s(c)}{k-1} \geq (w-1 - \frac{D}{r-k}) \log \binom{r}{k-1}$. This bound is at most $w \log k$ greater than the sum $\sum_{c|k < s(c) \leq r} \log \binom{s(c)}{k-1}$. Finally, a simple calculation shows that $\frac{1}{2} \sum_c \max(0, n - (2w-1)s(c)) = \frac{nD}{2r}$.

Thus we get the following lower bound on $\sum_c F(s(c))$:

$$n - w(k + \log k) + \min_{0 \leq D \leq (w-1)(r-k)} \left(\left(w - 1 - \frac{D}{r-k} \right) \log \binom{r}{k-1} + \frac{nD}{2r} + \log \binom{rw+D}{k-1} \right).$$

Since this is a concave function it is minimized either at $D = 0$ or $D = (w-1)(r-k)$. This yields the following lower bound on the worst-case number of comparisons required to solve the k -selection problem when $k \leq r$:

$$\frac{(w+1)n}{2} - w(k + \log k) - \frac{w^3}{8} + \min \left((w-1) \log \binom{r}{k-1} + \log \binom{rw}{k-1}, \frac{n(r-k)(w-1)}{2r} + \log \binom{n-(w-1)k}{k-1} \right). \quad \square$$

4.2.2 Lower bounds in the randomized query model

We now prove lower bounds on the number of queries used by randomized k -selection algorithms. We conjecture that the randomized algorithm for finding the minimal elements which we give in

the proof of Theorem 12 essentially achieves the lower bound. However, the lower bound we prove here is a factor 2 different from this upper bound.

We consider a distribution $D(n, w)$ on partial orders of width w over a set $P = \{x_1, \dots, x_n\}$. The distribution $D(n, w)$ is defined as follows:

- The support of $D(n, w)$ is the set of partial orders consisting of w chains, where any two elements from different chains are incomparable.
- Each element belongs independently to one of the w chains with equal probability.
- The linear order on each chain is chosen uniformly.

Theorem 18. *The expected query complexity of any algorithm solving the k -selection problem is at least*

$$\frac{w+3}{4}n - wk + w \left(1 - \exp\left(-\frac{n}{8w}\right)\right) \left(\log\left(\frac{n/(2w)}{k-1}\right)\right).$$

Proof. In order to provide a lower bound on the number of queries, we provide a lower bound on the number of queries of incomparable elements and then use the classical bound to bound the number of queries of comparable elements.

First we note that for each element a , the algorithm must make either at least one query where a is comparable to some other element b , or at least $w - 1$ queries where a is incomparable to all elements queried. (The latter may suffice in cases where a is the unique element of a chain and it is compared to all minimal elements of all other chains.)

We let $Y_t(i)$ denote the number of queries involving x_i before the first query for which the response is that x_i is comparable to an element. Also for each of the chains C_1, \dots, C_w we denote by Z_α the number of comparisons involving two elements from the same chain.

Letting T denote the total number of queries before the algorithm terminates, we obtain:

$$\mathbf{E}[T] \geq \sum_{i=1}^n \frac{1}{2} \mathbf{E}[Y_T(i)] + \sum_{\alpha=1}^w \mathbf{E}[Z_\alpha].$$

We claim that for all $1 \leq i \leq n$ we have $\mathbf{E}[Y_T(i)] \geq \frac{w-1}{2}$. This follows by conditioning on the chains that all other elements but x_i belong to. With probability $1/w$, the first query will give a comparison; with probability $1/w$, the second query, etc.

On the other hand, by the classical lower bounds we have for each $1 \leq \alpha \leq w$ that

$$Z_\alpha \geq |C_\alpha| - k + \log\left(\frac{|C_\alpha|}{k-1}\right)$$

Taking expected value we obtain

$$\mathbf{E}[Z_\alpha] \geq \frac{n}{w} - k + \mathbf{E}\left[\log\left(\frac{|C_\alpha|}{k-1}\right)\right].$$

A rough bound on the previous expression may be obtained by using the fact that by standard Chernoff bounds, except with probability $\exp(-\frac{n}{8w})$, it holds that C_α is of size at least $n/(2w)$. Therefore

$$\mathbf{E}\left[\log\left(\frac{|C_\alpha|}{k-1}\right)\right] \geq \left(1 - \exp\left(-\frac{n}{8w}\right)\right) \log\left(\frac{n/(2w)}{k-1}\right).$$

Summing all of the expressions above, we obtain

$$\frac{(w-1)n}{4} + w \left(\frac{n}{w} - k \right) + \left(1 - \exp \left(-\frac{n}{8w} \right) \right) w \log \left(\frac{n/(2w)}{k-1} \right)$$

and simplifying gives the desired result. \square

5 Computing linear extensions and heights

In this section we consider two problems that are closely related to the problem of determining a partial order: given a poset, compute a linear extension, and compute the heights of all elements.

A total order $(P, >)$ is a *linear extension* of a partial order (P, \succ) if, for any two elements x and y , $x \succ y$ implies $x > y$. We give a randomized algorithm that, given a set P of n elements and access to an oracle for a poset (P, \succ) of width at most w , computes a linear extension of (P, \succ) with expected total complexity $O(n(\log n + w))$. We give another randomized algorithm that, on the same input, determines the height of every element of (P, \succ) with expected total complexity $O(nw \log n)$.

The algorithms are analogous to Quicksort, and are based on a *ternary* search tree, an extension of the well-known binary search tree for maintaining elements of a linear order. A ternary search tree for (P, \succ) , consists of a root, a left subtree, a middle subtree and a right subtree. The root contains an element $x \in P$ and the left, middle, and right subtrees are ternary search trees for the restrictions of (P, \succ) to the sets $\{y \mid x \succ y\}$, $\{y \mid x \not\succ y\}$ and $\{y \mid y \succ x\}$, respectively. The ternary search tree for the empty poset consists of a single empty node. A randomized algorithm to construct a ternary search tree for (P, \succ) assigns a random element of P to the root, compares each of the $n - 1$ other elements to the element at the root to determine the sets associated with the three children of the root, and then, recursively, constructs a ternary search tree for each of these three sets.

Define the weight of an internal node x of a ternary search tree as the total number of internal nodes in its three subtrees, and the weight of a ternary search tree as the sum of the weights of all internal nodes. Then the number of queries required to construct a ternary search tree is exactly the weight of the tree.

Theorem 19. *The expected weight of a ternary search tree for any poset of size n and width w is $O(n(\log n + w))$.*

Proof Sketch. Consider the path from the root to a given element x . The number of edges in this path from a parent to a middle subtree is at most w , and the expected number of edges from a parent to a left or right subtree is $O(\log n)$ since, at every step along the path, the probability is at least $1/2$ that the sizes of the left and right subtrees differ by at most a factor of 3. It follows that the expected contribution of any element to the weight of the ternary search tree is $w + O(\log n)$. \square

Once a ternary search tree for a poset has been constructed, a linear extension can be constructed by a single depth-first traversal of the tree. If x is the element at the root, then the linear extension is the concatenation of the linear extensions of the following four subsets, corresponding to the node and its three subtrees: $\{y \mid x \succ y\}$, $\{x\}$, $\{y \mid x \not\succ y\}$ and $\{y \mid y \succ x\}$. The corollary below follows.

Corollary 20. *There is a randomized algorithm of expected total complexity $O(n(\log n + w))$ for computing a linear extension of a poset.*

Let $h(x) = h$ be the height of element x in (P, \succ) . Given a linear extension $x_n > \dots > x_2 > x_1$, it is easy to compute $h(x)$ for each element x by binary search, using the following observation: Let $S(i, h) = \{x_j \mid j \leq i, h(x_j) = h\}$ be the set of elements of index at most i in the linear extension and of height h in (P, \succ) . Then $|S(i, h)| \leq w$ (as the elements of $S(i, h)$ are pairwise incomparable), and $h(x_{i+1}) > h$ if and only if there exists $x \in S(i, h)$ such that $x_{i+1} \succ x$. Thus, given the sets $S(i, h)$, for all h , we can determine $h(x_{i+1})$ and the sets $S(i+1, h)$, for all h , in time $O(w \log i)$ using binary search. This yields:

Corollary 21. *Given a linear extension, there is a deterministic algorithm with total complexity $O(wn \log n)$ to compute the heights of all elements of a partial order of size n and width w . Combining this algorithm with the above algorithm for computing a linear extension, there is a randomized algorithm to determine the heights of all elements with expected total complexity $O(wn \log n)$.*

6 Variants of the poset model

In this section, we discuss sorting in two variants of the poset model that occur when different restrictions are relaxed. First, we consider posets for which a bound on the width is not known in advance. Second, we allow the irreflexivity condition to be relaxed, which leads to transitive relations. We show that with relatively little overhead in complexity, sorting in either case reduces to the problem of sorting posets.

6.1 Unknown width

Recall from Section 3 that $N_w(n)$ is the number of posets of width at most w on n elements.

Claim 22. *Given a set P of n elements and access to an oracle for poset $\mathcal{P} = (P, \succ)$ of unknown width w , there is an algorithm that sorts P using at most $2 \log w (\log N_{2w}(n) + 4wn) = \Theta(n \log w (\log n + w))$ queries, and there is an efficient algorithm that sorts P using at most $8nw \log w \log(n/(2w))$ queries with total complexity $O(nw^2 \log w \log(n/w))$.*

Proof. We use an alternate version of ENTROPYSORT that returns FAIL if it cannot insert an element (while maintaining a decomposition of the given width) and an alternate version of POSET-MERGESORT that returns FAIL if the PEELING algorithm cannot reduce the size of the decomposition to the given width. The first algorithm of the claim is, for $i = 1, 2, \dots$, to run the alternate version of algorithm ENTROPYSORT on input set P , the oracle, and width upper bound 2^i , until the algorithm returns without failing. The second algorithm is analogous but uses the alternate version of POSET-MERGESORT. The claim follows from Theorems 6 and 9, and from the fact that we reach an upper bound of at most $2w$ on the width of \mathcal{P} in $\log w$ rounds. \square

6.2 Transitive relations

A partial order is a particular kind of transitive relation. In fact, our results generalize to the case of arbitrary transitive relations (which are not necessarily irreflexive) and are therefore relevant to a broader set of applications. Formally, a transitive relation is a pair (P, \triangleright) , where P is a set of elements and $\triangleright \subseteq P \times P$ is transitive. The *width* of a transitive relation is defined to be the maximum size of a set of mutually incomparable elements. We say that a poset (P, \succ) is *induced* by a transitive relation (P, \triangleright) if $\succ \subseteq \triangleright$. A poset (P, \succ) is *minimally induced* by (P, \triangleright) if for any relation $(x, y) \in \triangleright \setminus \succ$, the pair $(P, \succ \cup (x, y))$ is not a valid partial order, i.e. its corresponding graph contains a directed cycle.

We require the following lemma, bounding the width of a minimally induced poset.

Lemma 23. *Let (P, \succ) be a poset minimally induced by the transitive relation (P, \succeq) . Then the width of (P, \succ) is equal to the width of (P, \succeq) .*

Proof. Suppose otherwise, that is, suppose that there is a pair of distinct elements $x, y \in P$ such that $x \not\succeq y$ with respect to the partial order (P, \succ) , but x and y have some relation in (P, \succeq) . Without loss of generality, suppose that $x \succeq y$; it may be simultaneously true that $y \succeq x$. First, we note that $(P, \succ \cup (x, y))$ is a valid partial order; if it were not, i.e. if the addition of (x, y) introduced a cycle, then it would be the case that $y \succ x$, which is a contradiction to their incomparability. However, the poset $(P, \succ \cup (x, y))$ is also induced by (P, \succeq) , which contradicts the assumption that (P, \succ) is minimally induced. \square

We denote by \mathcal{O}_\succ an oracle for a poset (P, \succ) and by \mathcal{O}_\succeq an oracle for a transitive relation (P, \succeq) . In the following claim, we assume that the poset sorting algorithm outputs a chain decomposition (such as a CHAINMERGE); if it does not, the total complexity of the algorithm for sorting a transitive relation increases a bit, but not its query complexity.

Claim 24. *Suppose there is an algorithm \mathcal{A} that, given a set P of n elements, access to an oracle \mathcal{O}_\succ for a poset $\mathcal{P} = (P, \succ)$, and an upper bound of w on the width of \mathcal{P} , sorts P using $f(n, w)$ queries and $g(n, w)$ total complexity. Then there is an algorithm \mathcal{B} that, given P , w , and access to an oracle \mathcal{O}_\succeq for a transitive relation (P, \succeq) of width at most w , sorts P using $f(n, w) + 2nw$ queries and $g(n, w) + O(nw)$ total complexity.*

Proof. Given an oracle \mathcal{O}_\succeq for the transitive relation (P, \succeq) , we define a special poset oracle \mathcal{O} that runs as follows: Given a query $q(x, y)$, the oracle \mathcal{O} first checks if the relation between x and y can be inferred by transitivity and irreflexivity from previous responses. If so, it outputs the appropriate inferred response; otherwise, it forwards the query to the oracle \mathcal{O}_\succeq . The oracle \mathcal{O} outputs the response of \mathcal{O}_\succeq except if both $x \succeq y$ and $y \succeq x$; in this case, \mathcal{O} outputs whichever relation is consistent with the partial order determined by previous responses (if both relations are consistent, then it arbitrarily outputs one of the two). By definition, the responses of \mathcal{O} are consistent with a partial order induced by (P, \succeq) .

The first step of algorithm \mathcal{B} is to run algorithm \mathcal{A} on input P and w , giving \mathcal{A} access to the special oracle \mathcal{O} , which \mathcal{B} simulates using its access to \mathcal{O}_\succeq . Since \mathcal{A} completely sorts its input, it reconstructs a poset induced by (P, \succeq) via \mathcal{O} that has a maximal set of relations. That is, there is a poset $\mathcal{P} = (P, \succ)$ minimally induced by (P, \succeq) such that the responses of \mathcal{O} to the sequence of queries made by \mathcal{A} are indistinguishable from the responses of \mathcal{O}_\succ to the same sequence of queries. Since \mathcal{P} has the same width as (P, \succeq) , it is valid to give \mathcal{A} the upper bound of w . Hence, \mathcal{A} sorts \mathcal{P} and outputs some chain decomposition $\mathcal{C} = \{C_1, \dots, C_q\}$ of \mathcal{P} such that $q \leq w$.

The second step of algorithm \mathcal{B} is to make a sequence of queries to the oracle \mathcal{O}_\succeq to recover the relations in $\succeq \setminus \succ$. It is similar to building a CHAINMERGE data structure: for all i, j , $1 \leq i, j \leq q$, for every element $x \in C_i$, we store the index of x in chain C_i and the index of the largest element $y \in C_j$ such that $x \succeq y$. An analysis similar to the one for CHAINMERGE (see Section 2) shows that it takes at most $2nq$ queries to the oracle \mathcal{O}_\succeq and $O(nq)$ total complexity to find all the indices. The relation in (P, \succeq) between any pair of elements can then be looked up in constant time. \square

References

- [1] G. Brightwell. “Balanced Pairs in Partial Orders,” *Discrete Mathematics* 201(1–3): 25–52, 1999.
- [2] G. Brightwell and S. Goodall. “The Number of Partial Orders of Fixed Width,” *Order* 20(4): 333–345, 2003.

- [3] G. Brightwell and P. Winkler. “Counting Linear Extensions is #P-Complete,” *STOC* 1991.
- [4] T. M. Cover and J. A. Thomas. *Elements of information theory*. New York: John Wiley & Sons Inc., 1991.
- [5] U. Faigle and Gy. Turán. “Sorting and Recognition Problems for Ordered Sets,” *SIAM J. Comput.* 17(1): 100–113, 1988.
- [6] L. R., Jr., Ford and D. R. Fulkerson. *Flows in Networks*. Princeton University Press, 1962.
- [7] M. Fredman. “How good is the information theory bound in sorting?” *Theor. Comput. Sci.* 1(4): 355-361, 1976.
- [8] F. Fussenegger and H. N. Gabow. “A Counting Approach to Lower Bounds for Selection Problems,” *Journal of the ACM* 26(2): 227–238, 1979.
- [9] J. Kahn and J. H. Kim. “Entropy and Sorting,” *STOC*, 178 – 187, 1992.
- [10] J. Kahn and M. Saks. “Balancing poset extensions,” *Order* 1(2): 113–126, 1984.
- [11] S. S. Kislitsyn, “A finite partially ordered set and its corresponding set of permutations,” *Matematicheskie Zametki* 4(5): 511–518, 1968.
- [12] D. Knuth. *The Art of Computer Programming: Sorting and Searching*, Massachusetts: Addison-Wesley, 1998.
- [13] N. Linial. “The Information theoretic bound is good for merging,” *SIAM J. Comput. SICOMP* 13(4): 795-801, 1984.
- [14] M. E. J. Newman, *SIAM Review* 45: 167–256, 2003.
- [15] K. Onak and P. Parys. Generalization of Binary Search: Searching in Trees and Forest-Like Partial Orders. *FOCS* 2006.
- [16] W. Trotter and S. Felsner, “Balancing pairs in partially ordered sets,” *Combinatorics, Paul Erdos is Eighty I*: 145–157, 1993.